University of North Georgia
Department of Mathematics

Instructor: Berhanu Kidane

Course: College Algebra Math 1111

Text Book: For this course we use the free e – book by Stitz and Zeager with link:

http://www.stitz-zeager.com/szca07042013.pdf

Tutorials and Practice Exercises

- http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/index.htm
- http://www.mathwarehouse.com/algebra/
- http://www.ixl.com/math/algebra-2
- http://www.ixl.com/math/precalculus
- http://www.ltcconline.net/greenl/java/index.html

For more free supportive educational resources consult the syllabus

Functions and Relations (Page 20)

Objectives: By the end of this chapter students should be able to:

- **Identify relations**
- **Define a relation**
- **Define a function and find the domain and range of a function**
- **Identify graphs of functions and sketch graphs of functions**
- **Describe the different transformations of functions, and sketch graphs using transformation of functions**
- **Identify one – to – one functions**

Introduction to Relations and Their Graphs(Page 20 – 28)

Relations

Important Ideas **Set** and **Ordered Pairs**

Definition (A set): A set is a collection of well-ordered objects.

Examples: a) The set of students in this class

b) The set of natural less than 11 $= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c) $\boldsymbol{E}$ $=$ The set of even natural numbers

$= \{2, 4, 6, 8, 10, . \ . \ .\}$

$= \{\, x : x \text{ is an even natural number} \,\}$

Definition (Ordered Pairs):

A pair which is written in the form $(\boldsymbol{a}, \boldsymbol{b})$ is called an **ordered** pair. In the pair $(\boldsymbol{a}, \boldsymbol{b})$**,** $\boldsymbol{a}$ is called **first** or $\boldsymbol{x}$ **coordinate** (or entry) and **b** is called **second** or **y** **coordinate** (or entry).

Note: $(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{c}, \boldsymbol{d})$ if and only if $\boldsymbol{a} = \boldsymbol{c}$ and $\boldsymbol{b} = \boldsymbol{d}$.

Definition (Relation)**:** A relation is a set of ordered pairs.

Example 1:

a) $\boldsymbol{R} = \{\, (\mathbf{2}, -\mathbf{5})\,,\ (\mathbf{5},\ \mathbf{6}),\ (-\mathbf{7}, \mathbf{6}),\ (\mathbf{2}, \mathbf{7}),\ (-\mathbf{7},\ \mathbf{3})\}$

b) $\boldsymbol{F} = \{\, (\boldsymbol{a}, \boldsymbol{b})\,,\ (\mathbf{3},\ \mathbf{4}), (\boldsymbol{c}, \boldsymbol{d})\,\}$

Example 2:

a) $\boldsymbol{R} = \{\, (\boldsymbol{x}, \boldsymbol{y}) : \ \boldsymbol{x} + \boldsymbol{y} > \mathbf{0},\ \ \boldsymbol{x} - \boldsymbol{y} < \mathbf{1},\ \ \boldsymbol{y} \leq \mathbf{2}\}$

b) $\boldsymbol{F} = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{y} > \boldsymbol{x}^{\mathbf{2}} \text{ and } \boldsymbol{y} \leq \mathbf{4}\}$

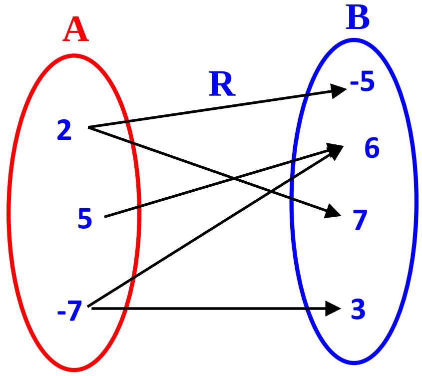
A **relation** can also be **represented** by:

i) A Venn-diagram, ii) A graph, or iii) An Equation

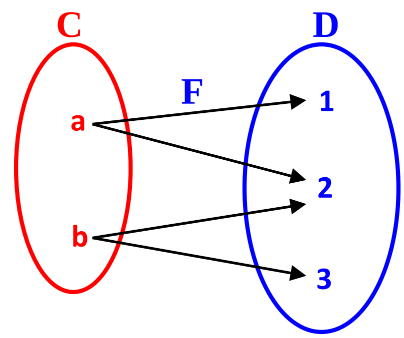
i) Venn-diagrams of relations

Example 1:

a)



b)



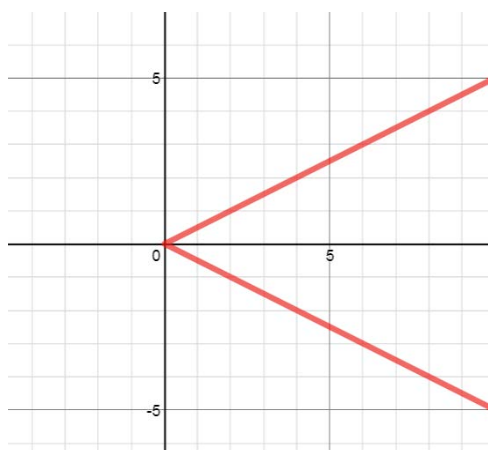
R is a relation from **A** into **B**

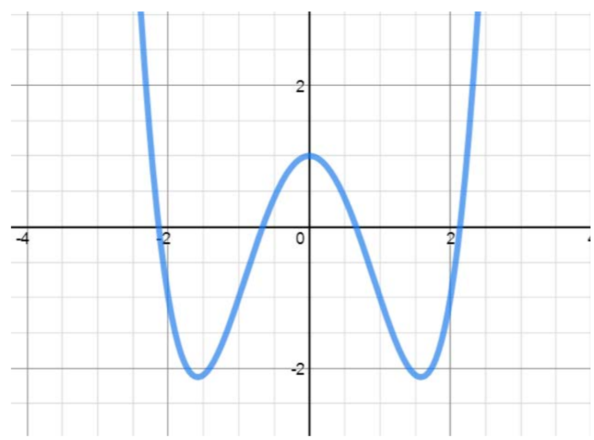
Using the set notation **R** $=$

F is a relation from **C** into **D**

Using the set notation **F** $=$

ii) Graphs of relations





iii) Equations defining relations

Example 2:

a) $y = x^2 - 5x + 9$

b) $|y| = x + 1$

c) $y \geq -2x - 4, y > x + 1$ *and* $y \leq 2$

d) $2x - 6y > 3$ *and* $x + y < 1$

e) $y^2 = x$

f) $x^2 + y^2 = 1$

Example YouTube videos

- Relations and functions https://www.youtube.com/watch?v=Uz0MtFlLD-k
- Functions Part 1: https://www.youtube.com/watch?v=3IjfebJgPP8

Homework

Practice problems: relation-function-worksheet pdf , shared class files

Page 29 – 32: 1-56 odd numbers (Stitz and Zeager Book)

Functions and Their Graphs (page 43)

Important Ideas:

Definition of a Function, Venn diagrams, Graphs,
Domain, Range, Functional Values,
Functional Notations, Equations Defining Functions,
Vertical Line Test

Definition 1: A **function** is a **relation** for which **each element** in the **domain corresponds** to **exactly one element** in the **range**. In other words, **every** x **can only** be **paired with one** y.

Or Equivalently

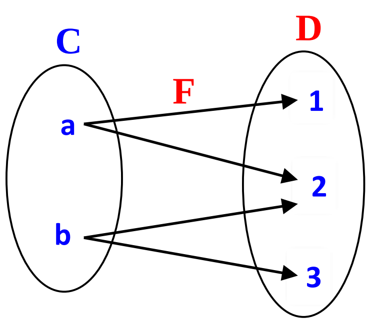
Definition 2: A **relationship** between **two variables**, typically x and y, is called a **function** if there is a rule that assigns to **each value** of x one and only one value of y. We then say that y is a **function** of x.

Note: For functions **the same** x **value cannot** have **2 different y values**.

Example 1: a) $\mathbf{R} = \{(\mathbf{2},\mathbf{3}),\ (\mathbf{1},\mathbf{5}),\ (\mathbf{0},\mathbf{4})\}$, is a **function**, why?

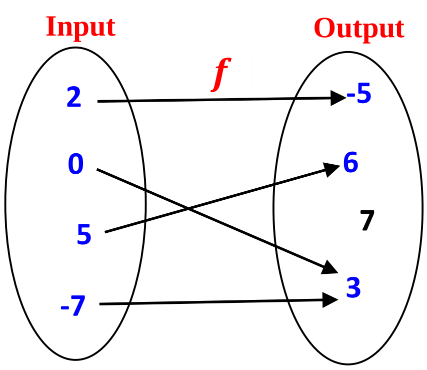
b) $\mathbf{R} = \{(\mathbf{2},\mathbf{3}),\ (\mathbf{2},\mathbf{5}),\ (\mathbf{0},\mathbf{4}),\ (\mathbf{3},\mathbf{4})\}$, is **not** a **function**, why?

Example 2: Venn diagrams



F is not a function, why?

Example 3:



f is a **function,** every x is **paired** to **exactly** one **y.**

Example YouTube videos

- Introduction to functions: https://www.youtube.com/watch?v=VhokQhjl5t0&list=PLDECCD8714DD4B0A8
- Function 2: https://www.youtube.com/watch?v=XEblO51pF5I
- Function 3: https://www.youtube.com/watch?v=5fcRSie63Hs

Domain and Range

Definition: Let f be a function.

- The **set** of all **first entries** is called the **DOMAIN** of the function f.
- The **set** of all **second entries** is called the **RANGE** of the function f.

Examples 4: Domain and range

a) $f = \{ (1,1),\ (-1,1),\ (2,4),\ (3,\ 9) \}$

Domain of $f = \{1,\ -1,\ 2,\ 3\}$, **Range of** $f = \{1,\ 4,\ 9\}$

b) $g = \{ (1,\ 4),\ (2,4),\ (3,5), (6,10) \}$

Domain of g = **Range of** g =

Example YouTube videos

- The domain of a function: https://www.youtube.com/watch?v=U-k5N1WPk4g
- Domain and range of a function given a formula: https://www.youtube.com/watch?v=za0QJRZ-yQ4
- Domain and range of a function: https://www.youtube.com/watch?v=O0uUVH8dRiU

Homework

Page 49 – 51: 1-50 odd numbers (Stitz and Zeager Book)

The Values of a Function and Functional Notation

By the **value of a function** we mean the **value of y**.

Functions are often denoted by the letters ***f, F, g*** *and* ***G*** etc.

Note: If f is a function, then for each **number** x in its **domain** the corresponding **image** in the **range** is **designated** by the symbol $f(x)$ and read as " f *of* x" or as "f *at* x". We **refer** to $f(x)$ as **the value** of f *at* x, *or* the **output corresponding** to x, or the **image of** x. **Note that** $f(x)$ **does not mean** f *times* x.

Note in functions:

Inputs or Pre-images are **1st** or $x - entries$, **1st** or $x - coordinates$ or $x - values$

Outputs or Images are **2nd** or $y - entries$, **2nd** or $y - coordinates$ or $y - values$

Example 1: *Read* each symbol.

a) $g(x)$; "g of x" b) $f(2)$; "f of 2" c) $g(-1)$; "g of -1"

d) $f(x^2 - 1)$; "f of $x^2 - 1$" e) $f(g(x))$; "f of g of x"

Functional Notation (page 61)

Examples: Let $y = x^2 + 1$; write $f(x)$ for the value **y.**

Then we call $f(x) = x^2 + 1$ the functional notation for $y = x^2 + 1$

Similarly: $f(x) = 3x - 5$ is a functional notation for $y = 3x - 5$; and

$f(x) = x^2$ is a **functional notation** for $y = x^2$ **and so on.**

Example 2: Finding functional values

Let $f(x) = 2x + 1$. **What is** $f(3)$, $f(-5)$, **and** $f(A)$?

Solution: Note, f maps x **to 2 times** x **plus 1**

$f(3) = 2(3) + 1 = 7$

$f(-5) = 2(-5) + 1 = -9$

$f(A) = 2(A) + 1 = 2A + 1$

Example 3: Let $y = 1 - x^3$. What is the value of **y** when

a) $x = 0$? b) $x = -1$? c) $x = q$? d) $x = -q$?

Example 4: If $h(x) = -2x + 1$, then

a) $h(x^3) =$

b) $h(x + 5) =$

c) $h(10) =$

Example 5: For $f(x) = x^2 + 3x + 1$, evaluate the following:

a) $f(0)$ b) $f(2)$

c) $f(-x)$ d) $f(x + 1)$

Example YouTube videos

- Evaluating Functions: https://www.youtube.com/watch?v=E9YEUQR9NAU
- Evaluating functions 2: https://www.youtube.com/watch?v=3i4MVwChSZc
- Functions (4): https://www.youtube.com/watch?v=rbt51hXmzig

Homework

Practice problems: Evaluating Functions pdf file shared class files

Evaluating-functions-worksheet shared class files

Page 63 – 68: 1-62 odd numbers, 63 – 68 odd numbers (Stitz and Zeager Book)

Graphs of Functions

The Graph of a Function

If f is a function with domain **A**, then the graph of f is the set of ordered pairs $\{(x, f(x)) | x \in A\}$ **plotted** in the **coordinate plane**. In other word the graph of f is the graph of the equation $y = f(x)$.

Example 1: Sketch the graph of the following functions

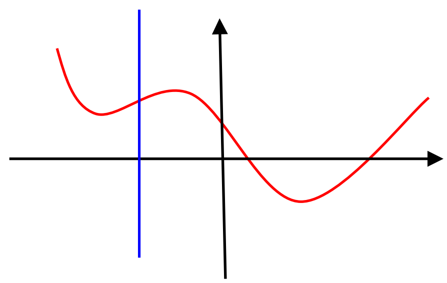
a) $y = x^2$ b) $y = 2x + 1$ **c)** $f(x) = x^3$

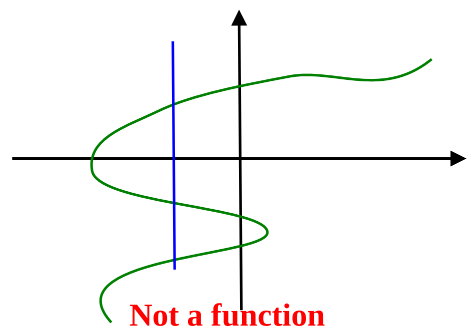
Example YouTube videos

- Graphing a Basic Function: https://www.youtube.com/watch?v=2-dUHLHeyTY

Vertical Line Test

A set of points in the xy - plane is the **graph of a function** if and only if **every vertical line intersects** the graph **in at most one point**.





Example YouTube videos

- Graphical relations and functions: https://www.youtube.com/watch?v=qGmJ4F3b5W8

Equations Defining Functions

To **determine** whether an **equation** in x and **y** **defines** a **function**, **solve** the **equation** for **y**. If we get only **one equation** expressed in terms of x, then the original equation is a function

Examples:

a) Show that $2y - 4x = 6$ defines a function.

Solve for **y** to get $y = 2x + 3$

The last equation shows that every x is paired to exactly one **y**.

b) Show that $y^2 = x$ is **not** a function**.** S**olving** for *y* **gives** $y = \pm\sqrt{x}$

If $x = 1$, then $y = \pm 1$, that is $x = 1$ **corresponds** to **two y values**

Thus, the equation $y^2 = x$ **does not** define a function.

c) $y = 2x + 1$ is a function of x since each **x-value,** input, results in only **1 y-value**.

d) $|y| = x$ is **not** a function of x**,** since $x = 9$ corresponds to both **y = 9** *and* **y = -9**.

e) $y = x^2$ is a function of x since each x-value, input, results in only 1 y-value, output.

f) Show that $x^2 + y^2 = 1$ is **not** a function

The Zeros of a Function

Definition: Let f be a function if $f(r) = 0$ for number r, then r is called the **zero of** f.

Example 1: Find the zeros of $f(x) = x^2 - 1$

Solution: To find the **zeros** of f we set $f(x) = 0$ and **solve for** *x*

$x^2 - 1 = 0$, which gives $x = -1$ *or* $x = 1$.

Thus **-1** and **1** are the zeros of the function $f(x) = x^2 - 1$.

Practice Problems Factors and Zeros pdf files Shared class files

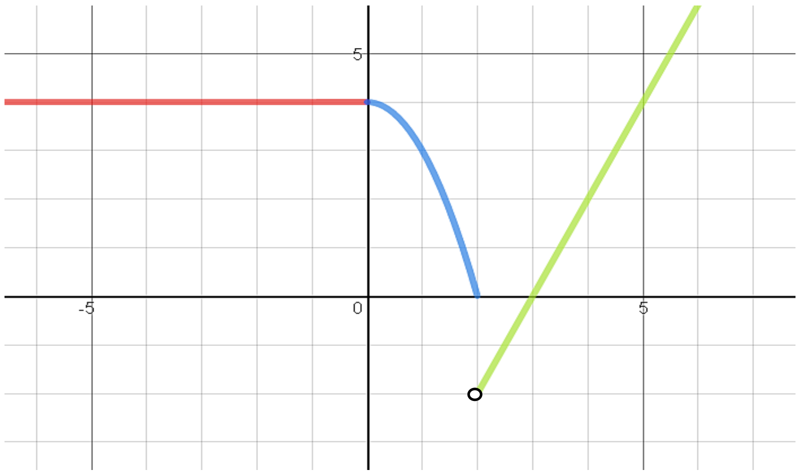
Example YouTube videos

- What is a function? https://www.youtube.com/watch?v=kvGsIo1TmsM

Piecewise Defined Functions

Piece-wise functions are formed by more than one function. Each function is defined for a specific set of values (intervals).

Example 1: Let $f(x) = \begin{cases} 4, & for\ x \leq 0 \\ 4 - x^2, & for\ 0 < x \leq 2 \\ 2x - 6, & for\ x > 2 \end{cases}$ Find $f(0)$, $f(-2), f(1), and\ f(5)$



Example 2: Amazon charges **$4** for an order under **$35** but provides free shipping for orders of **$35** or more. The cost ***C*** of **an order is a function** of the total **price** ***x*** of the books purchased, given by

$$C(x) = \begin{cases} x + 4, & if\ x < 35 \\ x, & if\ x \geq 35 \end{cases}$$

Example 3: In a certain state the **maximum speed** permitted on freeway is **65m/h**, and the **minimum** is **40m/h**. The **fine F** for violating these limits is **$15** for **every mile above** the **maximum** or **below** the **minimum**

a) If ***x*** the **speed** at which you are driving, then the fine function **F** is piecewise and given by the:

$$F(x) = \begin{cases} 15(40 - x), & if\ 0 < x < 40, \\ 0 & if\ 40 \leq x \leq 65 \\ 15(x - 65) & if\ x > 65, \end{cases}$$

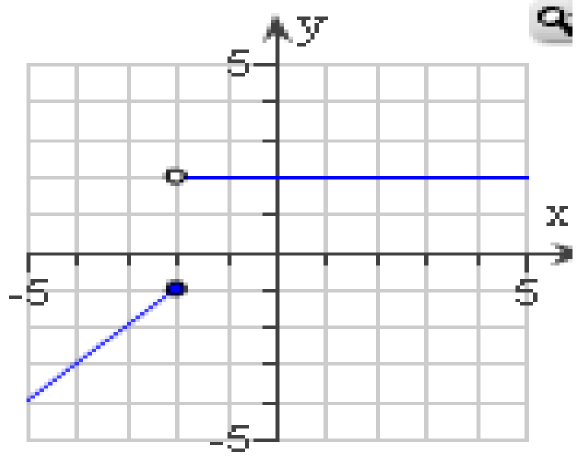
b) Find **F(30), F(50),** and **F(75)**

Example 4: Evaluate each for the following piece-wise function. $f(x) =$

$$\begin{cases} x^2, & if\ x < -1 \\ x + 1, & if\ -1 \leq x < 3 \\ 4, & if\ x \geq 3 \end{cases}$$

a) $f(-3)$ **b)** $f(0)$ **c)** $f(5)$

Example 5: Let $f(x) = \begin{cases} x+1, & if\ x \le -2 \\ 2, & if\ x > -2 \end{cases}$



Find:

a) $f(-7) =$

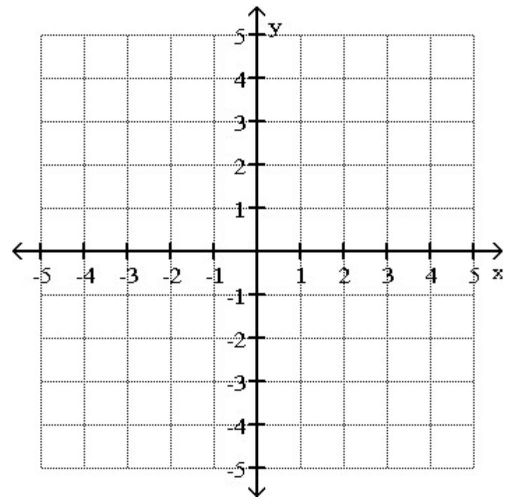
b) $f(-2) =$

c) $f(0) =$

d) $f(10) =$

Example 4: Sketch the graph of the piece-wise function, find the domain, range and find the values of the function at the given points.

$f(x) = \begin{cases} x+3, & if\ x \le -2 \\ 3, & if\ x > -2 \end{cases}$



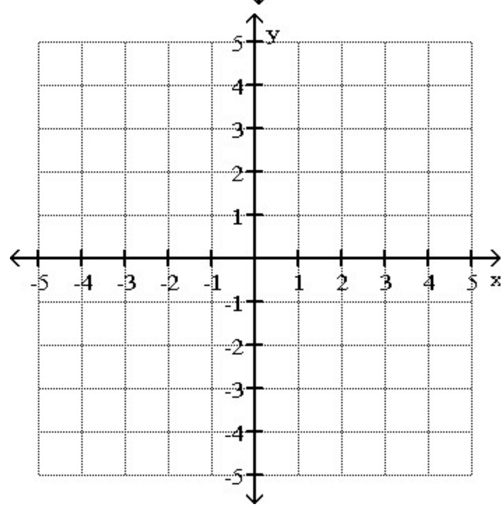
a) $f(-3) =$

b) $f(-2) =$

c) $f(110) =$

Example 5: Graph the piecewise function f

$$f(x) = \begin{cases} -3 - x, & if\ x \le -2 \\ 2x, & if\ -2 < x \le 2 \\ x^2 - 4x + 3, & if\ x > 2 \end{cases}$$



Find

a) $f(0)$

b) $f(-2)$

c) $f(10)$

OER West Texas A&M Tutorial 30: Introduction to Functions

Homework Exercise 1.6.2: page 107 #1 – 20 (Stitz and Zeager Book)

Example YouTube videos

- Piecewise function formula from graph: https://www.youtube.com/watch?v=tedzsRH0Jas
- Graphing piecewise function: https://www.youtube.com/watch?v=PQiXRrT_14o
- evaluate a piecewise function: https://www.youtube.com/watch?v=hg2HR9zJFq4

Difference Quotient and Average Rate

Difference Quotient (Page 79)

Let f be a function the **difference quotient** of f is defined as $\frac{f(x+h)-f(x)}{h}$

Example 4: Find the **difference quotient**, simplify your result.

a) $f(x) = x^2 + 2x + 1$

b) $f(x) = 2x - 3$

c) $f(x) = x^3 + 2x + 1$

d) $f(x) = \frac{x^2+2x-4}{x}$

Example 4: Reading: Example 1.5.2, page 79 – 81;

Solution: a) $f(x) = x^2 + 2x + 1$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2+2(x+h)+1-(x^2+2x+1)}{h}$$

$$= \frac{x^2+2xh+h^2+2x+2h+1-x^2-2x-1}{h}$$

$$= \frac{2xh+h^2+2h}{h}$$

$$= 2x + h + 2$$

Homework: Exercise 1.5.1 page 84: #21 – 45 (Stitz and Zeager Book)

Average Rates of Change

Definition: The average rate of change of $f(x)$ with respect to x for a function f as x changes from a *to* b is defined by $\frac{f(b)-f(a)}{b-a}$

Example: Find the **average rate** of change for the following.

a) $f(x) = x^2 - 2x + 1$ between $x = 0$ and $x = 3$

b) $y = \sqrt{x}$ between $x = 1$ and $x = 4$

c) $y = x^3$ between $x = -2$ and $x = 2$

Example: YouTube Video

- The difference quotient of a function: https://www.youtube.com/watch?v=WOjTJTHxsYc
- Average rate of change: https://www.youtube.com/watch?v=f4MYCepzLyQ

Basic Graphs

1) A constant function
2) The identity function
3) The absolute value function
4) $y = x^2$; The Square function
5) The square root function
6) The cubic and cube root functions
7) The reciprocal function
8) The greatest integer function

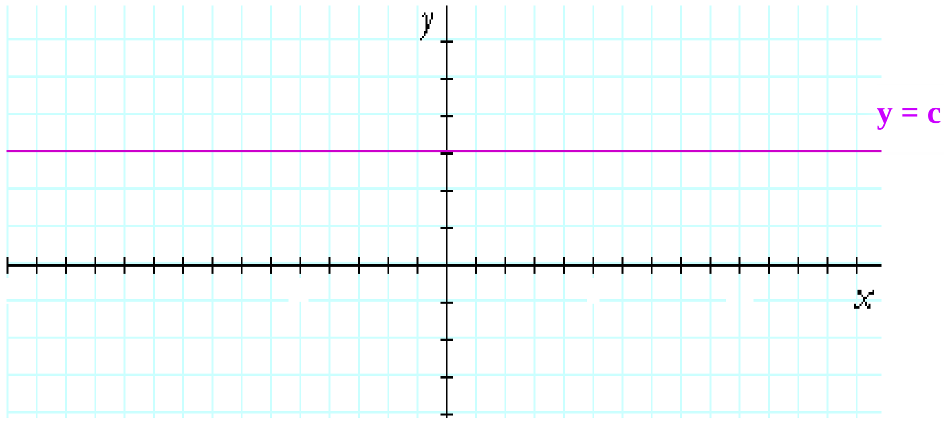
OER http://www.themathpage.com/aprecalc/graph-of-parabola.htm#absolute

a) A Constant Function

A constant function has the general form

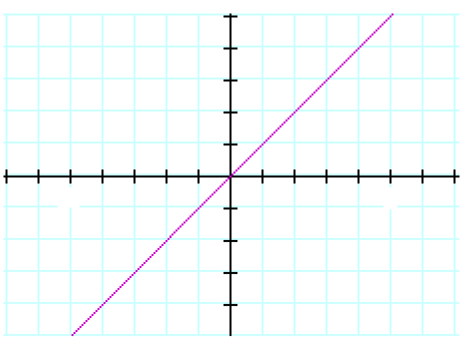
$y = f(x) = c$**,** where c is a constant, that is, a number

For example of the constant function $y = f(x) = 3$ Its graph is a straight line parallel to the x-axis.



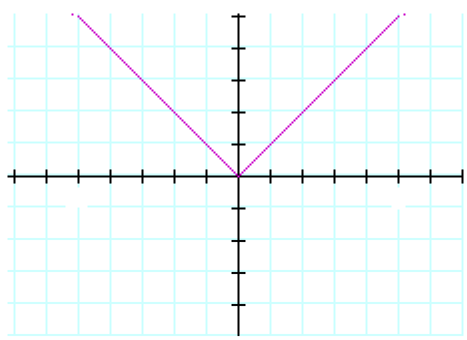
Question: Find the domains and ranges of the constant function

b) The Identity Function and the Absolute Value Function



$y = x$

The identity function

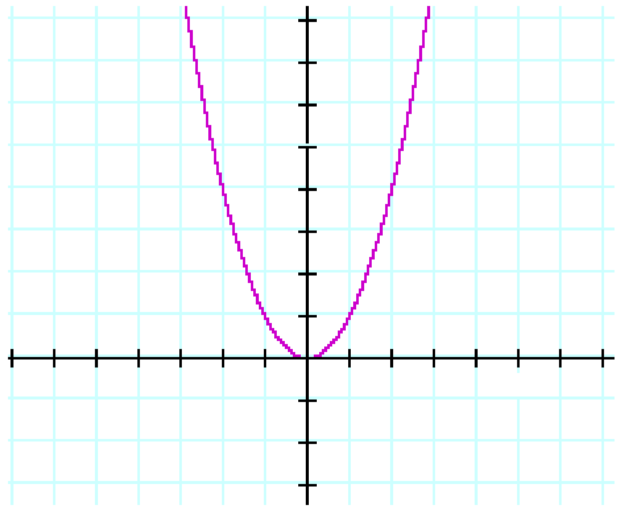


$y = |x|$

The absolute value function

Question: Find the domains and ranges of the identity and the absolute value functions.

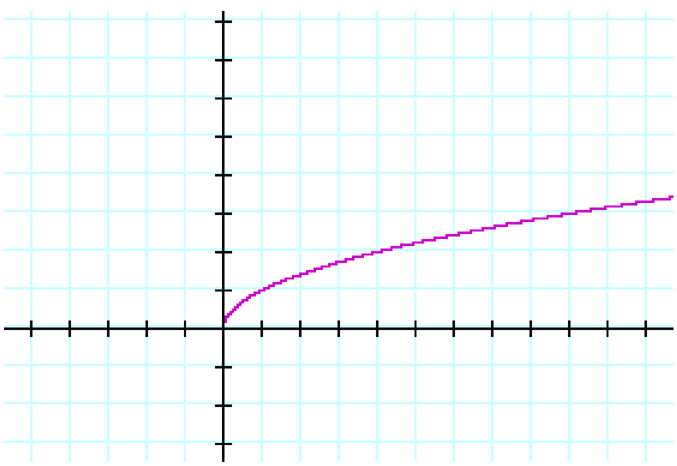
c) **A) $y = x^2$ the square function, Parabola**



$y = x^2$

A parabola

B) $y = \sqrt{x}$ The Square Root Function



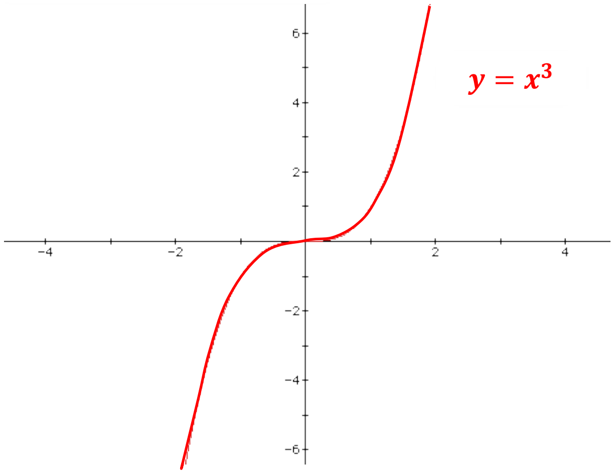
$y = \sqrt{x}$

The square root function

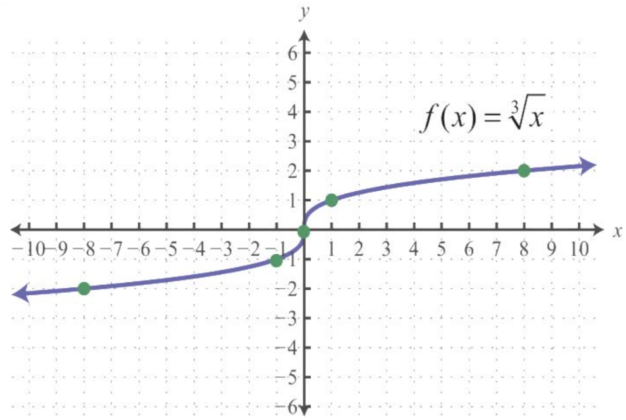
Question: Find the domains and ranges of the Parabola and the square root functions.

d) The Cubic and the Cube Root Functions

The Cubic Function



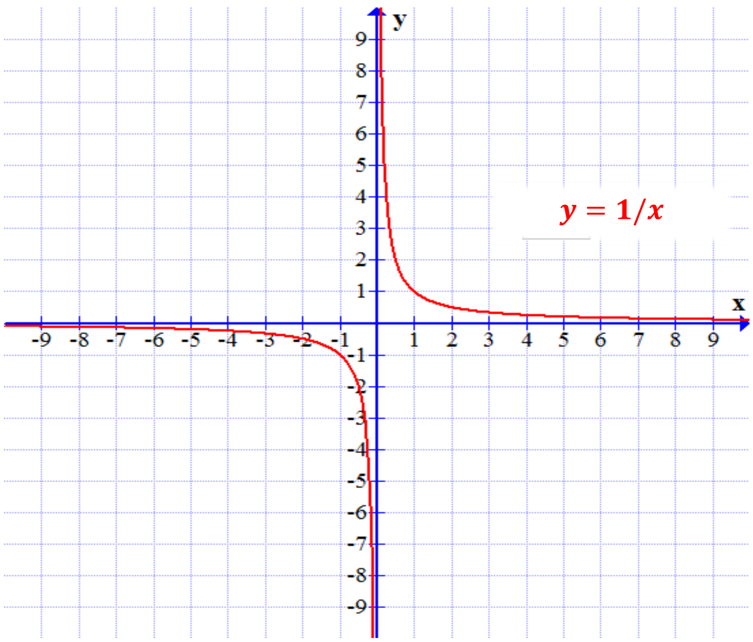
The Cube root Function



Question: Find the domain and range of the cubic and the cube root functions

e) The Reciprocal Function

The reciprocal function $y = f(x) = {}^{1}/_{x}$



Question: Find the domain and range of the cubic functions

f) The Greatest Integer function

The greatest integer function is denoted and defined by $y = f(x) = [x]$

$[x]$ Means the **greatest integer less than** or **equal** to x

Example: Let $f(x) = [x]$. Find the following values.

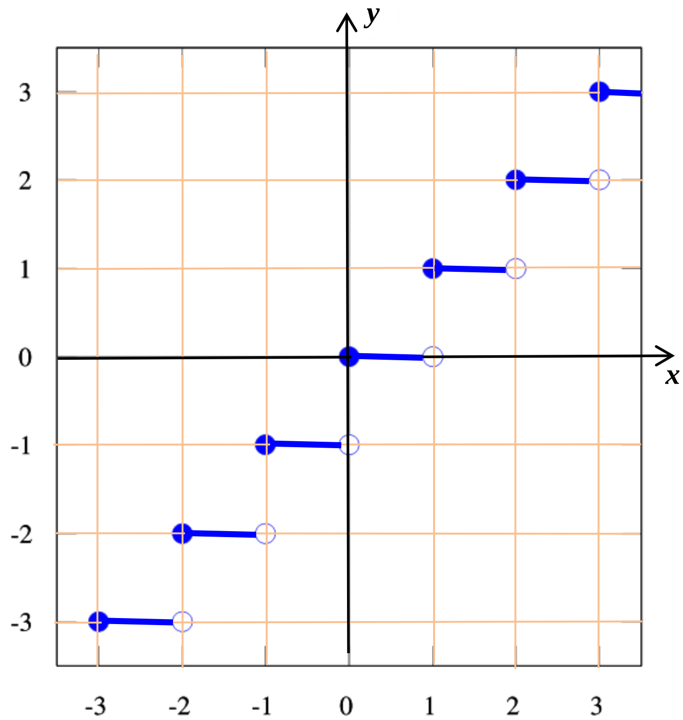
a) $f(0.5)$

b) $f(3.1)$

c) $f(-0.25)$

d) $f(-3)$

Graph of the Greatest Integer Function $y = [x]$



Question: Find the domain and range of the greatest integer functions

OER West Texas A&M Tutorial 31: Graphs of Functions, Part I

More on functions (Page 93)

- Increasing, decreasing and constant functions
- Even Odd Functions and symmetry
- Combination of functions
- Transformation and symmetry

Increasing, decreasing and Local Maximums and Minimums

Definition:

a) A function f is said to be an **increasing** function on an interval **I**,

if for all x_1 and x_2 in **I**, $x_1 < x_2$ implies that $f(x_1) < f(x_2)$.

- **Increasing**: where the function is **rising**.

  Trace the graph from left to right; where you go up is where the graph is increasing

b) A function f is said to be an **decreasing** function on an interval **I**,

if for all x_1 and x_2 in **I**, $x_1 < x_2$ implies that $f(x_1) > f(x_2)$.

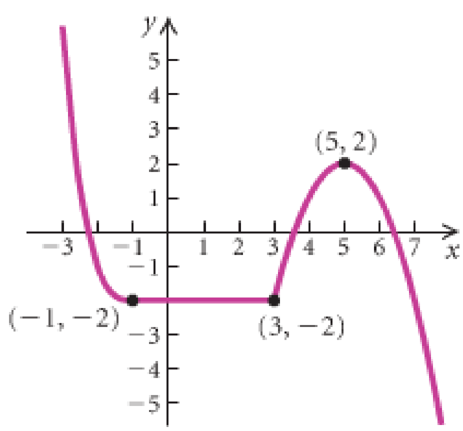
- **Decreasing**: where the function is **falling**.

  Trace the graph from left to right; where you go down is where the graph is decreasing

c) If the value of a function f **does not change** in an interval **I**, then f is **constant** on **I**

- **Constant:** where the function is horizontal

Example 1: Determine the intervals where the graph is increasing, decreasing, or constant.



Example 2: Find intervals where $a)\ f(x) = x^2 + 2,$ $b)\ f(x) = -x^2 - x$ and $c)\ f(x) = 1/x$ is:

a) Increasing
b) Decreasing
c) Constant

d) Local Minimum = __________
e) Local maximum = _________

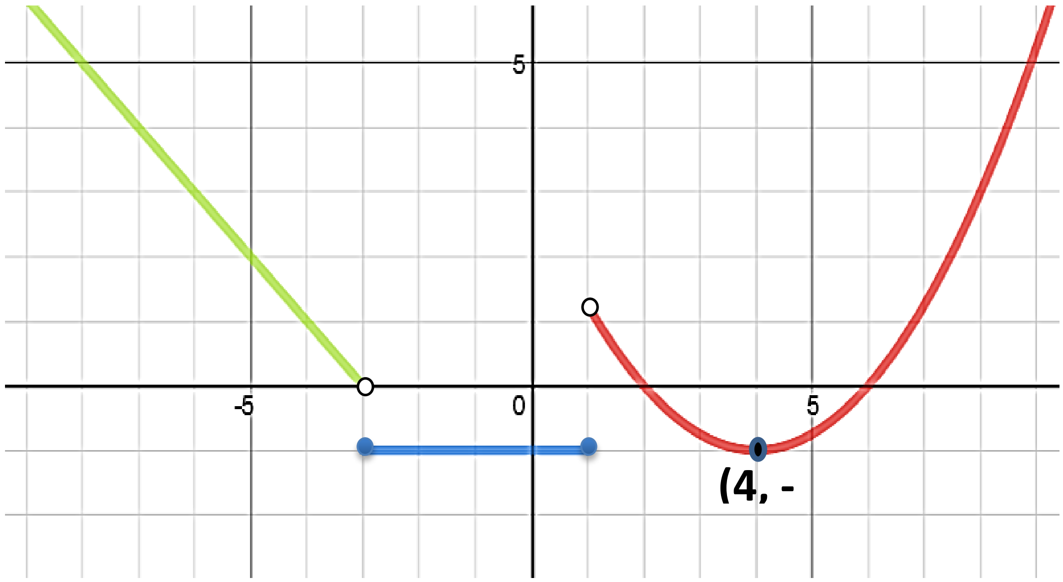
Example: YouTube Video

- Increasing/Decreasing, Local Maximums/Minimums: https://www.youtube.com/watch?v=aJuJOB6NTuc

OER West Texas A & M Tutorial 32: Graphs of Functions, Part II

Example 3: Find the interval where the function f is **increasing**, **decreasing** or a **constant**

$$f(x)=\begin{cases}\left(\frac{1}{2}x-2\right)^2-1 & if\ x>1\\ y=-1\quad if\ -3\leq x\leq 1\\ -x-3 & if x<-3\end{cases}$$



Even and Odd Functions and Symmetry:

Definition (even function)

A function f is even if $f(-x)=f(x)$ for all x in the domain of f

- An even function has graph that is **symmetric** with respect to (**wrt**) the **y-axis**

Definition (odd function)

A function f is odd if $f(-x)=-f(x)$ for all x in the domain of f

- An odd function has graph that is **symmetric** with respect to (**wrt**) the **origin**.

Example 1: Determine whether each of the functions is even, odd, or neither.

a) $f(x)=-3x^2+2x$

Solution: $f(x)=-3(-x)^2+2(-x)$

$=-3x^2 - 2x$

$=-(3x^2\ +2x)$

$=-f(x)$

Thus, the function is odd.

b) $f(x) = 3x^3 - 2x + 5$

Solution: $f(-x) = 3(-x)^3 - 2(-x) + 5$

$= \mathbf{-3x^3 + 2x + 5}$

$= -(\mathbf{3x^2 - 2x - 5}) \neq \boldsymbol{f(x)} \neq \boldsymbol{-f(x)}$

Thus, the function is **neither even nor odd**

c) $f(x) = 5x^4 + 2x^2 - 1$ (is even, show)

Example 2: Describe the following functions as **even, odd** or **neither** and **justify**

a) $f(x) = x^4 - x^3 + 12$ b) $f(x) = x^3 + 2x$ c) $f(x) = x^4 + x^3$

Intercepts and Symmetries

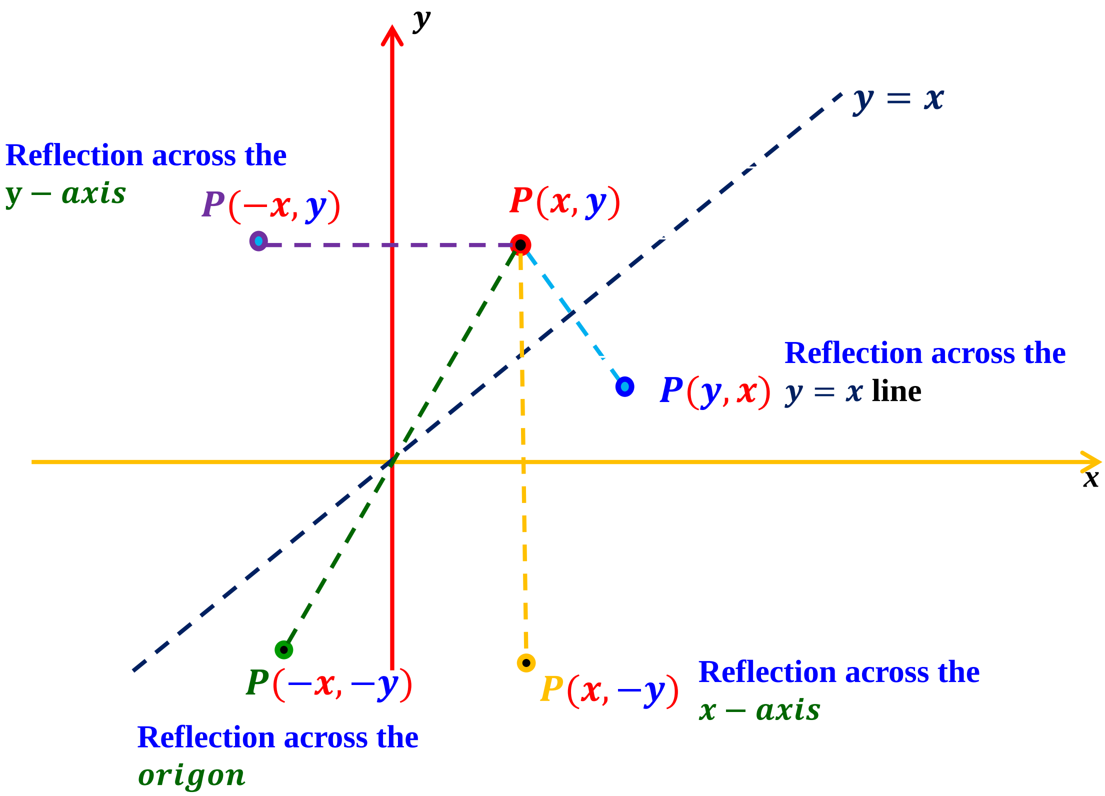
Objectives: By the end of this section you should be able to

- Find x and y intercepts
- Identify three types of symmetry: symmetry with respect to the y-axis, symmetry with respect to the origin, and symmetry with respect to the x-axis.
- Test equations for symmetry
- Read the domain and range from the graph

Symmetry

Given a point $\boldsymbol{P(x, y)}$

- $(\boldsymbol{x, -y})$ is a point of **symmetry** of the **point P** with respect to the **x-axis**
- $(\boldsymbol{-x, y})$ is a point of **symmetry** of the **point P** with respect to the **y-axis**
- $(\boldsymbol{-x, -y})$ is a point of **symmetry** of the **point P** with respect to the **origin**

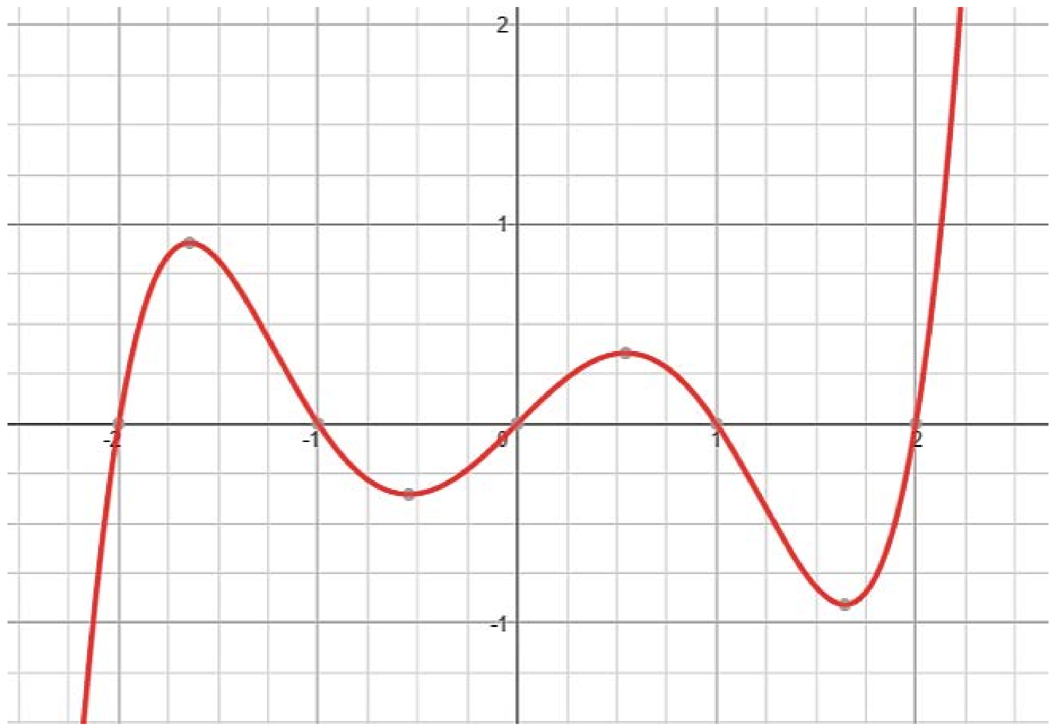


Intercepts: **x - intercept and y-intercept**

x - Intercepts are **points** (ordered pairs of numbers) where a **graph intersects** the **x - axis**.
y - Intercepts are **points** (ordered pairs of numbers) where a **graph intersects** the **y - axis**.

Note: At $\boldsymbol{x\ intercept\ y = 0}$ and at $\boldsymbol{y\ intercept\ x = 0}$

Example 4: Find the intercepts the local maximum and local minimum points form the graph



Example 5: Find the x and y intercepts:

a) $y = 2x - 3$ d) $y = x^2 - 1$ g) $y - 2xy + 2x = 1$

b) $2y + 4x = 6$ e) $9x^2 + 4y^2 = 36$

c) $y = x^2 - 5x + 6$ f) $y^2 = x^2 - 9$

Example: YouTube Videos

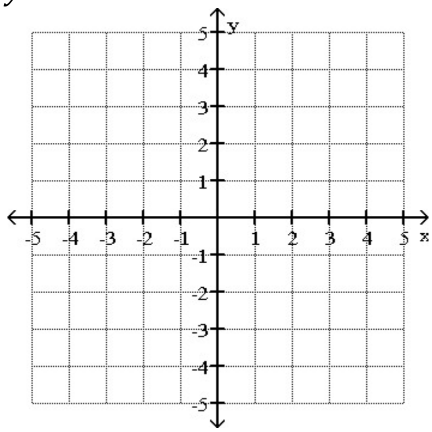
- Find x and y intercepts: https://www.youtube.com/watch?v=xGmef7IFc5w
- Finding intercepts: https://www.youtube.com/watch?v=405boztgZig

OER West Texas A & M Tutorial 32: Graphs of Functions, Part II

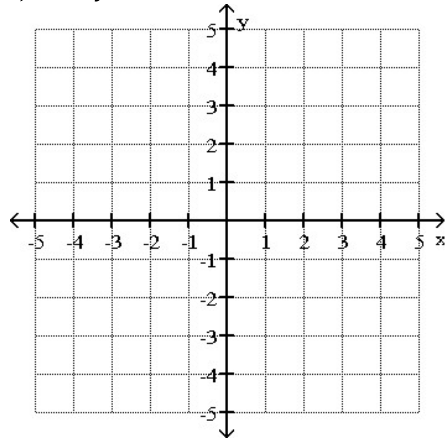
Homework: Exercise 1.6.2: page 107, #21 – 41 (Stitz and Zeager Book)

Example 6: Find the **intercepts** and **sketch** graphs.

a) $3x - 2y = 6$



b) $x + y = 0$



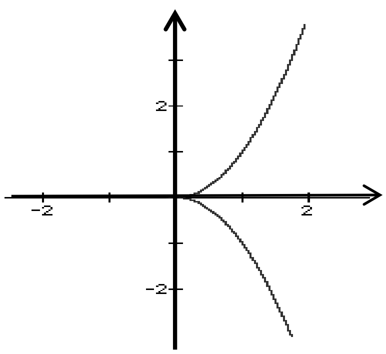
Symmetry

Symmetry with respect to **the x-axis, the y-axis, and the origin**

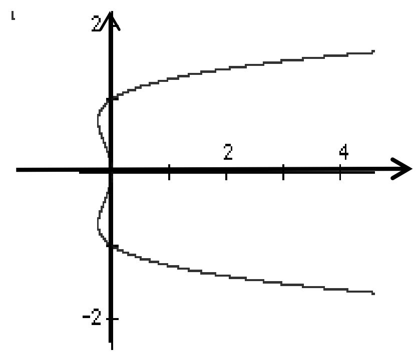
1) $\boldsymbol{x - axis}$ **Symmetry:** A graph is said to be **symmetric with respect to the x - axis if and only if** for every point (x, y) on the graph the point $(x, -y)$ is also on the graph.

Example1:

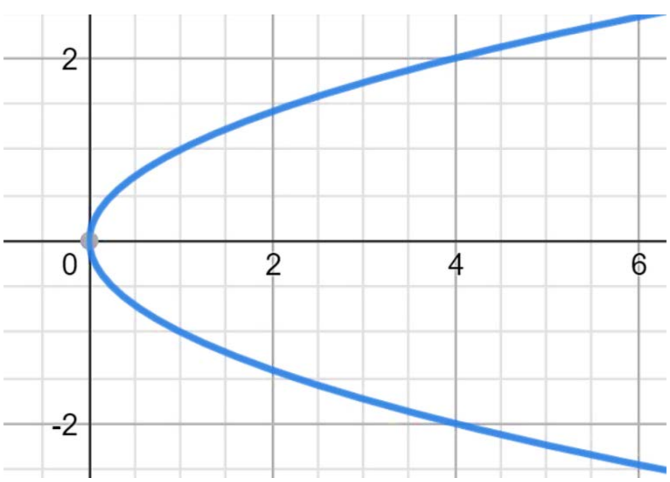
a)



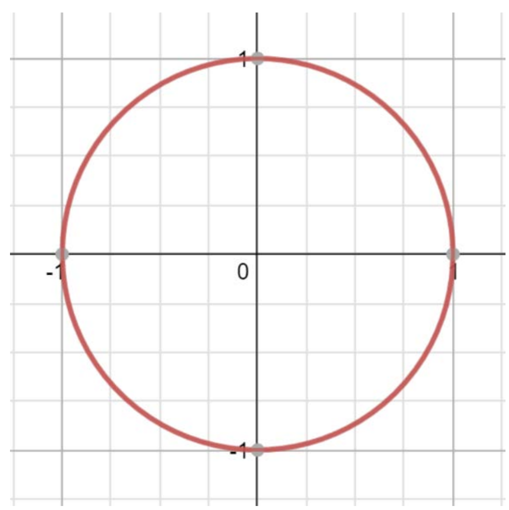
b)



Example 2: a) $y^2 = x$



b) Unit Circle $x^2 + y^2 = 1$



The Algebra of Functions (Page 76)

Basic Operations between two functions:

If f and g are functions and x is in the domain of each function, then we define the **sum**, **difference**, **product**, and **quotient** of f and g as follows

	Definition	Domain
Sum:	$(f+g)(x) = f(x) + g(x)$	$Domain\ of\ f \cap Domain\ of\ g$
Difference:	$(f-g)(x) = f(x) - g(x)$	$Domain\ of\ f \cap Domain\ of\ g$
Product:	$(f \times g)(x) = f(x) \times g(x)$	$Domain\ of\ f \cap Domain\ of\ g$
Quotient:	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$Domain\ of\ f \cap Domain\ of\ g$ **excluding** $\{x \mid g(x) \neq 0\}$

Example 1: Given $f(x) = x^2 - 3$ and $g(x) = 2x + 1$, find the following:

$a)\ (f+g)(x) = f(x) + g(x)$
$= (x^2 - 3) + (2x + 1)$
$= x^2 + 2x - 2$

$b)\ (f\,g)(x) = f(x)g(x)$
$= (x^2 - 3)(2x + 1)$
$= 2x^3 + x^2 - 6x - 3$

c) $(f - g)(x)$

$d)\ (f+g)(5) = f(5) + g(5)$
$= (5^2 - 3) + (2(5) + 1)$
$= (25 - 3) + (10 + 1)$
$= 22 + 11 = 33$

e) $(f\,g)(2)$ f) $\left(\frac{f}{g}\right)(x)$ g) $\left(\frac{f}{g}\right)\left(-\frac{1}{2}\right)$

Example 2: Find the domain of f, g, $f+g$, $f-g$, $f \times g$, $\frac{f}{g}$, $\frac{g}{f}$ where

$f(x) = x^2 - 3$ and $g(x) = 2x + 1$

Domain of f: $(-\infty, \infty)$

Domain of g: $(-\infty, \infty)$

Domain of $f + g$: $(-\infty, \infty)$

Domain of $f\,g$: $(-\infty, \infty)$

Domain of $\frac{f}{g}$: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$

Domain of $\frac{g}{f}$: $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$

Example 3: Let f be the function defined by

$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$

and let g be the function defined by

$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}$

Compute the indicated value if it exists.

a) $(f+g)(-3)$ b) $(f \div g)(-2)$ c) $(f-g)(2)$ d) $(g \div f)(-3)$ e) $(f \times g)(-1)$ f) $(f \circ g)(-3)$ g) $(g \circ f)(-1)$

Example: YouTube Video

- Function Operations and Composition of Functions: https://www.youtube.com/watch?v=llt_ewKc7l4

Example 4: a) Find the domain of f, g, $f+g$, $f-g$, fg, $\frac{f}{g}$, $\frac{g}{f}$ where

$f(x) = x + 2$ *and* $g(x) = \sqrt{x-1}$

b) Find $(f+g)(x)$, $(f-g)(x)$, $(f \times g)(x)$, $(f \div g)(x)$, and $(g \div f)(x)$.

Composite Functions (page 359)

Definition: The **composite function** $f \circ g$, the composition of f *and* g**,** is defined as

$(f \circ g)(x) = f(g(x))$, where x is in the domain of g and $g(x)$ is in the domain of f**.**

Example 5: Example 3: Given $f(x) = x^2 - 3$ and $g(x) = 2x + 1$, find

a) $(f \circ g)(x)$
b) $(f \circ g)(1)$
c) $(g \circ f)(-x)$
d) $(g \circ f)(-2)$
e) $(f \circ f \circ f)(1)$

Solution: a)

$$(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 - 3 = 4x^2 + 4x - 2$$

Decomposition of Functions

In decomposing a function we will make **two** functions out of the **given function**.

Example 6: Find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$.

a) Decompose the function $h(x) = (4 + 3x)^5$

Solution: a) We can write $h(x)$ as:

$h(x) = (f \circ g)(x) = f(g(x))$, where $g(x) = 4 + 3x$ and $f(x) = x^5$

b) Decompose the function $p(x) = \sqrt{x^2 + 4}$

c) Decompose the function $r(x) = e^{5x-3}$

d) Decompose the function $f(x) = \frac{3}{x^2 - 2x}$

Examples 5.1.1, 5.1.2, & 5.1.3: Homework, Reading Page 360 – 367

OER West Texas A&M Tutorial 30B: Operations with Functions

Example YouTube videos

- Evaluating composite functions example: https://www.youtube.com/watch?v=jlID_mIJXi4

Practice problems: **Function Operations pdf files Shared class files**
operations-with-functions pdf files Shared class files

Homework: Exercise 1.5.1 Page 84 #1 – 20 & #51 – 62
Exercise 5.1.1 page 369 # 1 – 40 & 44 – 55 (Stitz and Zeager Book)

Transformations of Functions

Transformations of functions we consider in this section are:

Translations and **Reflections; Vertical** and **Horizontal Shrinks** and **Stretches**

Translations

1) Vertical Translation: $y = f(x) \pm c, for\ c > 0$

The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted vertically c units up

The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted vertically c units down

2) Horizontal Translations: $y = f(x \pm c), for\ c > 0$

The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted horizontally c units to the right

The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted horizontally c units to the left.

Reflections

1) Across the x-axis:

The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ across the **x-axis**.

2) Across the y-axis:

The graph of $y = f(-x)$ is the **reflection** of the graph of $y = f(x)$ across the **y-axis**.

Stretches and Shrinks

1) Vertical Stretching and shrinking

To graph $y = cf(x)$:

- If $c > 1$, **stretch** the graph of $y = f(x)$ **vertically** by a **factor of c**
- If $0 < c < 1$, **shrink** the graph of $y = f(x)$ **vertically** by a **factor of c**

2) Horizontal Stretching and shrinking

To graph $y = f(cx)$:

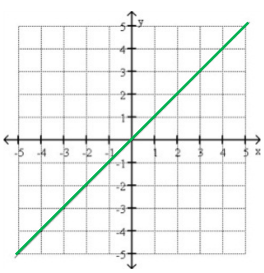
- If $c > 1$, **shrink** the graph of $y = f(x)$ **horizontally** by a **factor of $1/c$**
- If $0 < c < 1$, **stretch** the graph of $y = f(x)$ **horizontally** by a **factor of $1/c$**

Example: YouTube Video

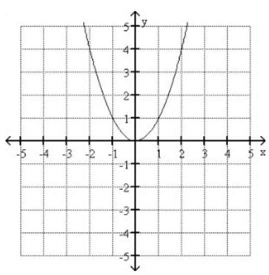
- Parent Functions and Transformations: https://www.youtube.com/watch?v=6h6fAd3Za1E
- Translations, Stretches, and Shrinks: https://www.youtube.com/watch?v=zmz1uamLXlI
- Stretching, Compressing, and Reflecting: https://www.youtube.com/watch?v=-UnIpuCbDjQ

Recall the basic graphs

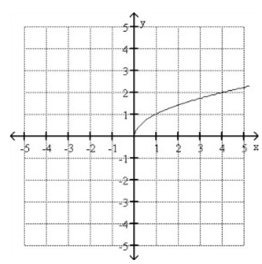
$f(x) = x$



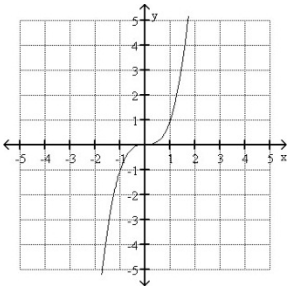
$f(x) = x^2$



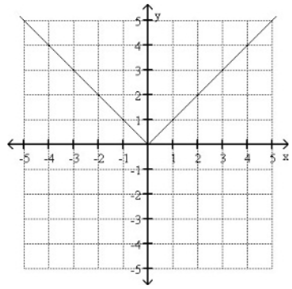
$f(x) = \sqrt{x}$



$f(x) = x^3$

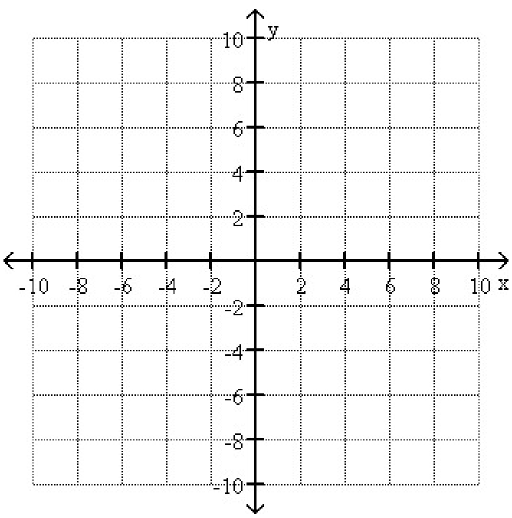


$f(x) = |\,x\,|$

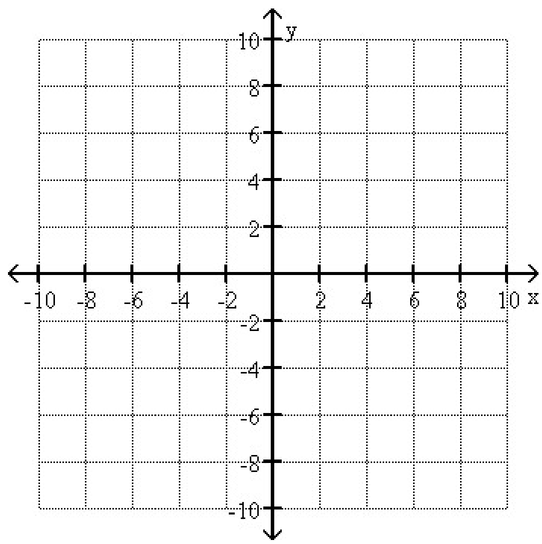


Example 1: Using the given information sketch the graph and give the equation.
Given $f(x) = x^2$

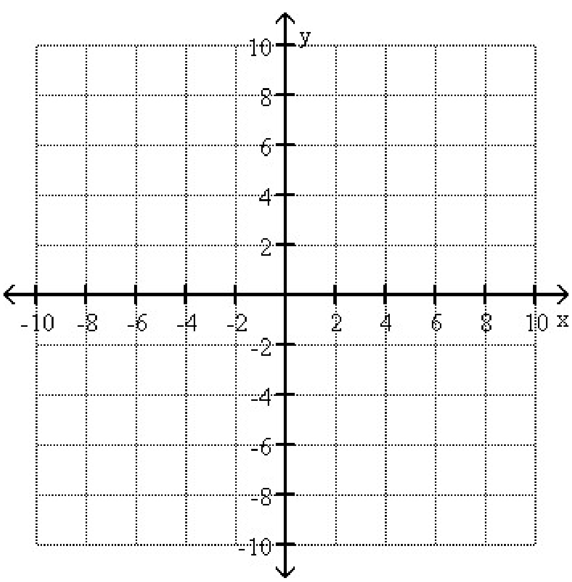
a) $f(x) = x^2 + 3$



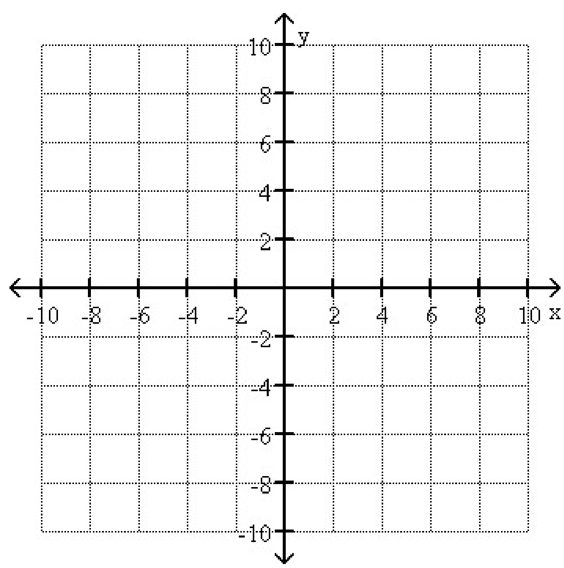
b) $f(x) = -(x+1)^2 + 2$



c) $f(x) = (x-1)^2 + 1$



d) The graph of $f(x) = x^2$, but **upside down**, and **shifted left 2 units**



Example 2: Given $y = \sqrt{x}$ sketch the graph or give the equation

a) The graph of $f(x) = \sqrt{x}$, but shifted left 4 units

b) The graph of $y = \sqrt{x-2}$

c) The graph of $y = -\sqrt{x} + 1$

Example 3: Using **reflection**, **horizontal** and **vertical shifts** and the graph of $y = |x|$ sketch

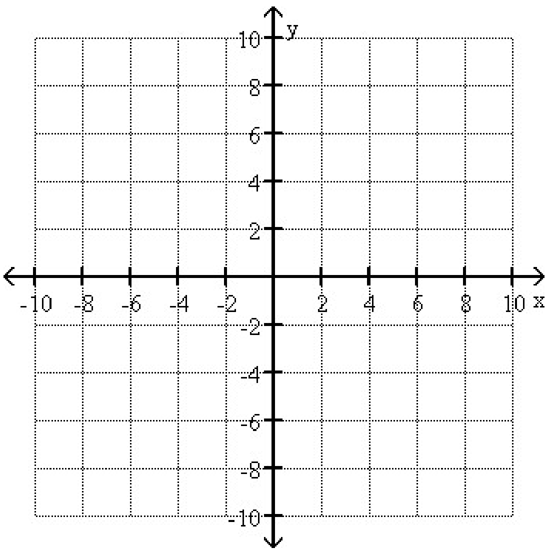
a) $f(x) = -|x|$

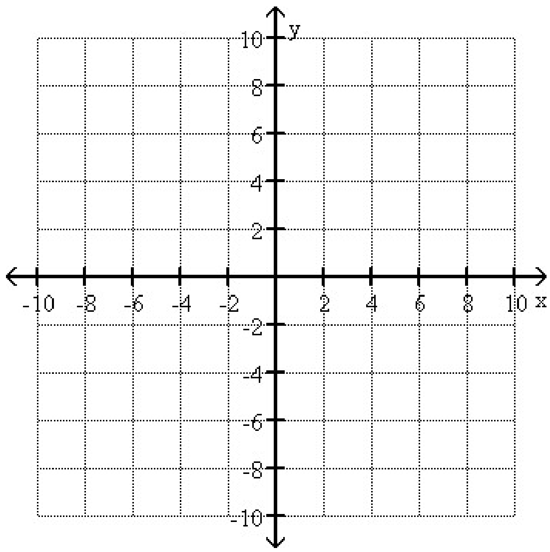
b) $f(x) = |x - 1| + 1$

c) $f(x) = |x - 2| - 1$

d) $f(x) = |x + 1| + 1$

e) $f(x) = -|x + 2| + 3$





Example 4: Using transformation sketch the graph of the following functions

a) $f(x) = |x|$ **Parent function**

b) $f(x) = x^2$ **Parent function**

c) $f(x) = |3x|$ horizontal shrink by a factor of 1/3

d) $f(x) = \left|\frac{1}{3}x\right|$ horizontal stretch by a factor of 1/3

e) $f(x) = 2x^2$ vertical stretch by a factor of 2

f) $f(x) = \frac{1}{2}x^2$ vertical shrink by a factor of 2

g) $f(x) = (2x)^2$ horizontal shrink by a factor of 1/2

h) $f(x) = \left(\frac{1}{2}x\right)^2$ horizontal stretch by a factor of 1/2

One – to – One Functions and Inverse Functions (Page 378)

Objectives: By the end of this section you should be able to

- Identify one – to – one functions
- Find the inverse of a function and domain and range of inverse functions
- Use the Horizontal line test
- Identify one – to – one functions from graphs
- Graph inverse functions
- State the inverse function property

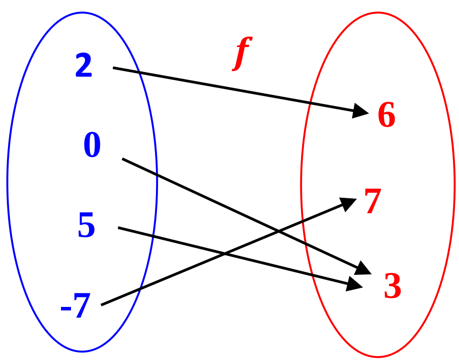
One – to – One Function

Definition:

- A function f is said to be a **one – to – one function if and only if** for every a, b in the domain of f $a \neq b$ **implies** that $f(a) \neq f(b)$. That is f is **one – to –one if and only if different inputs** always have **different outputs. Or equivalently**
- A function f is said to be a **one – to – one function** if every **y** in the **range** related to **exactly one** x in the **domain**. **Or equivalently**
- A function f is said to be a **one – to – one** if every **horizontal** line **intersects** the graph of f at **most once**. (**Horizontal Line Test**)

1. Venn Diagrams (Example 1)

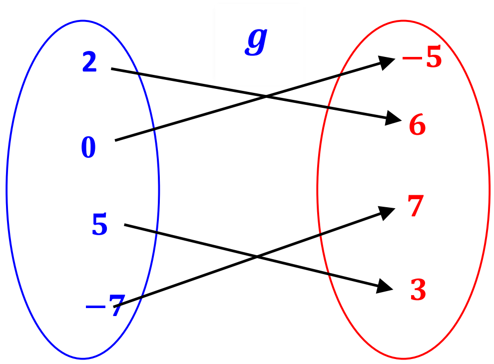
a)



$f = \{(2, 6), (0, 3), (5, 3), (-7, 7)\}$

f **is not** a **one – to – one function. Why?**

b)

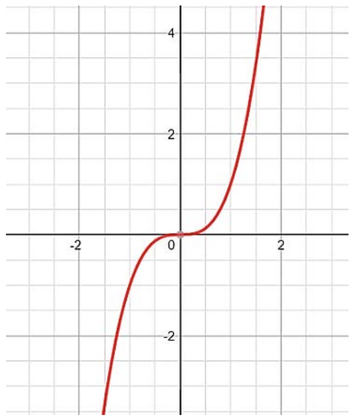


$g = \{(2, 6), (0, -5), (5, 3), (-7, 7)\}$

g is a **one – to – one function. Why?**

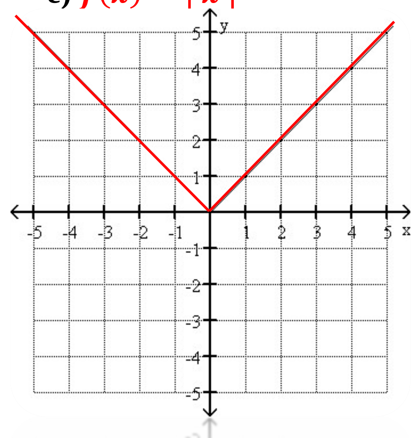
2. Graphs (Example 2)

d) $f(x) = x^3$



One – to – one, why?

e) $f(x) = |\ x\ |$



Not one – to – one, why?

Example 3: Verify the following functions are **one – to – one**.

a) $g = \{(2, 6), (0, -5), (5, 3), (-7, 7)\ \}$

b) $f(x) = 3x - 2$

c) $f(x) = \frac{1}{x}$

d) $g(x) = \sqrt{x}\ ,\ \ x \geq 0$

e) $h(x) = \ x^3$

f) $f(x) = (x - 1)^2 - 2,\ \ x\ \geq 1$

Example 5.2.1 page 382 reading

The Inverse of a Function

Definition: **The inverse** of a function f is a relation defined as the set $\{(y, x): whenever\ (x,\ \ y)\ belongs\ to\ f\ \}$

Example 4: Find the inverse

a) $f = \ \{(2,\ \ 6), (0, 3), (5, 3), (-7, 7)\}$

Solution: *Inverse of* $f = \{(6,\ \ 2), (3,\ \ 0), (3,\ \ 5), (7, -7)\ \}$

Note: The inverse of f is **not** a **function**. **Why**?

b) $g = \{(2, 6), (0, -5), (5, 3), (-7, 7)\ \}$

Solution: Inverse of $g = \{(6,\ \ 2), (-5,\ \ 0), (3,\ \ 5), (7, -7)\ \}$

Here: The Inverse of g is a function**. Why?**

Note: The inverse of a function may not always be a function

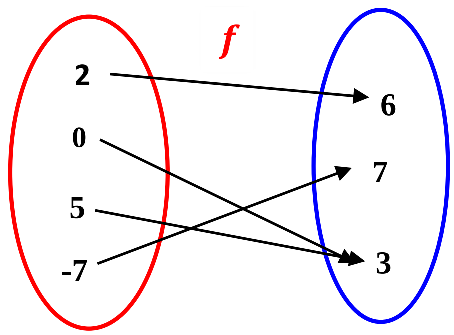
Question:

- When is the **inverse** of a function, a **function?**
- In other words, which function inverse gives **inverse function**?

To answer these questions consider the following Venn Diagrams

Example 5: Venn Diagrams

a)

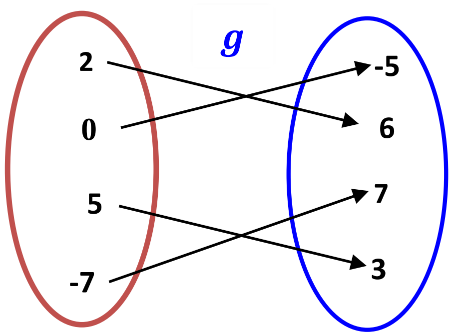


$f = \{(2,\ 6), (0, 3), (5, 3), (-7, 7)\}$

Inverse of f =

Is the inverse of f a function?

b)



$g = \{(2, 6), (0, -5), (5, 3), (-7, 7)\}$

Inverse of g =

Is the inverse of g a function?

Note: The inverse of a function f is a function if the original function is **one-to-one**

Inverse Functions

Definition: The inverse of a **one-to-one function** f , written as f^{-1}, is a function given by:

$$f^{-1} = \{(y,\ x): \text{whenever } (x, y) \text{ belongs to } f\}.$$

Note:

- If f is a **one – to – one function,** we call f^{-1} the **inverse function of** f
- If f takes x in to y the inverse function f^{-1} takes y in to x, that is; if $y = f(x)$, then $x = f^{-1}(y)$

Inverse function Property

Let f be a **one – to – one function** with **domain** A and **range** B. The inverse function f^{-1} has **domain** B and **range** A and **satisfies** the following **cancellation properties**:

$f^{-1}\left(f(x)\right) = x$ for every x in A

$f\left(f^{-1}(x)\right) = x$ for every x in B

Conversely any **function** f^{-1} **satisfying** these cancellation equations is the **inverse** of f

Finding the Inverse Functions from Equations

Procedures for finding the inverse function f^{-1}:

1) Write y for $f(x)$
2) Solve for x in 1)
3) 2) gives $x = f^{-1}(y)$
4) Finally, in 3) replace **y with** x; that is, write the equation in terms of x

Example 5.2.2 page 384 reading

Example 6: For each of the following functions find the inverse function

a) $g = \{(2,6),(0,-5),(5,3),(-7,7)\}$

b) $f(x) = 3x - 2$

c) $f(x) = \frac{1}{x}$

d) $g(x) = \sqrt{x}\,,\ x \geq 0$

e) $h(x) = x^3$

f) $f(x) = (x-1)^2 - 2,\ x \geq 1$

Solutions:

b) $f(x) = 3x - 5$**,** replace $f(x)$ with **y**

$y = 3x - 5$**,** solve for x

$x = \frac{1}{3}y - \frac{5}{3}$**, thus the inverse function is** $f^{-1}(y) = \frac{1}{3}y - \frac{5}{3}$

Finally replacing **y with** x we get

$f^{-1}(x) = \frac{1}{3}x - \frac{5}{3}$

d) $g(x) = \sqrt{x}\,,\ x \geq 0$**, replace** $g(x)$ **by y**

$y = \sqrt{x}\,,$ Solve for **y.** Note $x \geq 0$ implies $y \geq 0$

$x = y^2$**,** Squaring both sides

Thus, $f^{-1}(y) = y^2,\ y \geq 0$, replacing **y** ***with*** x we get

$f^{-1}(x) = x^2,\ x \geq 0$**,** the inverse function

f) $f(x) = (x-1)^2 - 2,\ x \geq 1,$ **replace** $f(x)$ with **y**

$y = (x-1)^2 - 2,\ x \geq 1$**, solve for** x

$(x-1)^2 = y + 2$ **,** which implies

$x - 1 = \pm\sqrt{y+2}$, since $x \geq 1$ **, i.e.** $x - 1 \geq 0$ we take the positive square root

$x - 1 = \sqrt{y+2}$, which gives

$x = 1 + \sqrt{y+2}$

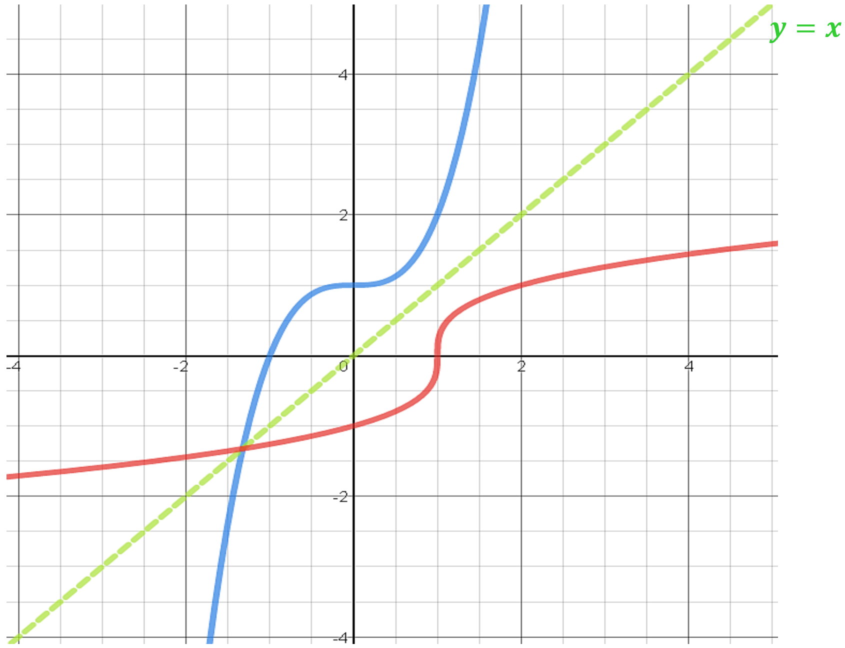
Thus, $f^{-1}(x) = 1 + \sqrt{x+2}\,, x \geq -2$**,** inverse function of f

Graphs of Inverse Functions

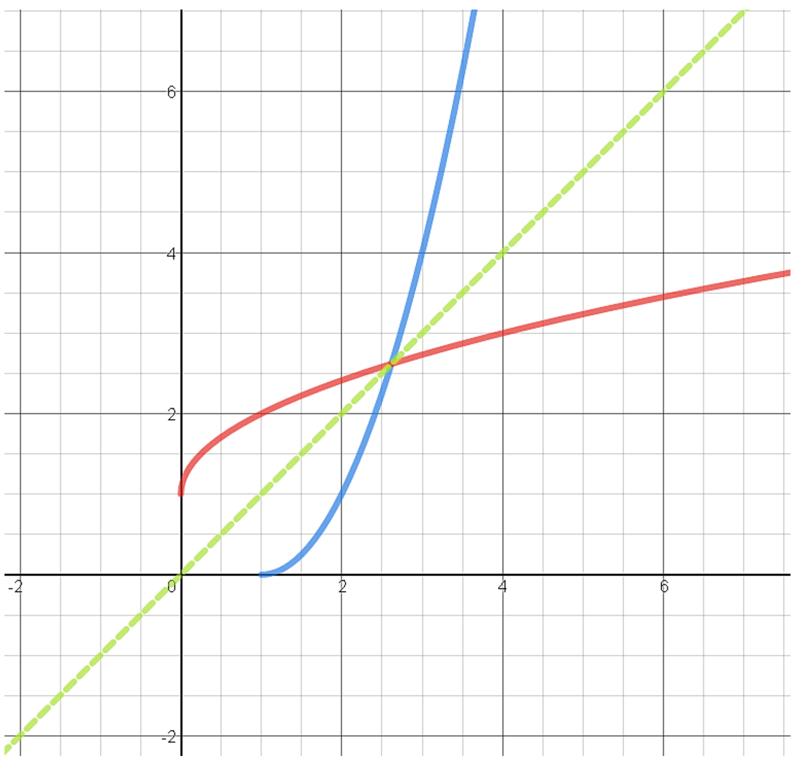
Recall: The reflection of a point $(a,\ b)$ about the line $y = x$ is the point $(b,\ a)$, but $(b,\ a)$ is a point in f^{-1} whenever $(a,\ b)$ is in f. Thus, the graph of f^{-1} is the **reflection** about the line $y = x$ of the graph of f.

Example 5.2.3 page 389 reading

Example 1: Graph of $f(x) = x^3 + 1$ and its inverse $f^{-1}(x) = \sqrt[3]{x-1}$. Graph of $f^{-1}$is the reflection of the graph of f across the line $y = x$



Example 2: Graph of $f(x) = (x-1)^2, x \geq 1$ and its inverse $f^{-1}(x) = \sqrt{x} + 1$



Example 3: For each of the following functions **sketch** the **inverse function**

g) $g = \{(2,6),(0,-5),(5,3),(-7,7)\}$

h) $f(x) = 3x - 2$

i) $f(x) = \frac{1}{x}$

j) $g(x) = \sqrt{x},\ x \geq 0$

k) $h(x) = x^3$

l) $f(x) = (x-1)^2 - 2,\ x \geq 1$

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Example: YouTube Video

- Introduction to function inverses: https://www.youtube.com/watch?v=W84lObmOp8M
- Function inverses example 1: https://www.youtube.com/watch?v=wSiamij_i_k
- Function inverses example 2: https://www.youtube.com/watch?v=W84lObmOp8M

Homework

Exercise 5.2.1 page 394 # 1 – 24 (Stitz and Zeager Book)

Practice Test

OER West Texas A&M Tutorial 33: Practice Test on Tutorials 25 - 32

More Practice problems Links

https://www.ixl.com/math/precalculus

Functions

1. **A.1**Domain and range
2. **A.2**Identify functions
3. **A.3**Linear functions
4. **A.4**Linear functions over unit intervals
5. **A.5**Evaluate functions
6. **A.6**Add, subtract, multiply, and divide functions
7. **A.7**Composition of functions
8. **A.8**Identify inverse functions
9. **A.9**Find values of inverse functions from tables
10. **A.10**Find values of inverse functions from graphs
11. **A.11**Find inverse functions and relations

Families of functions

1. **B.1**Translations of functions
2. **B.2**Reflections of functions
3. **B.3**Dilations of functions
4. **B.4**Transformations of functions
5. **B.5**Function transformation rules
6. **B.6**Describe function transformations

Quadratic functions

1. **C.1**Characteristics of quadratic functions
2. **C.2**Find the maximum or minimum value of a quadratic function
3. **C.3**Graph a quadratic function
4. **C.4**Match quadratic functions and graphs
5. **C.5**Solve a quadratic equation using square roots
6. **C.6**Solve a quadratic equation by factoring
7. **C.7**Solve a quadratic equation by completing the square
8. **C.8**Solve a quadratic equation using the quadratic formula
9. **C.9**Using the discriminant

Polynomials

1. **D.1**Divide polynomials using long division
2. **D.2**Divide polynomials using synthetic division
3. **D.3**Evaluate polynomials using synthetic division
4. **D.4**Write a polynomial from its roots
5. **D.5**Find the roots of factored polynomials
6. **D.6**Rational root theorem
7. **D.7**Complex conjugate theorem
8. **D.8**Conjugate root theorems
9. **D.9**Descartes' Rule of Signs
10. **D.10**Fundamental Theorem of Algebra
11. **D.11**Match polynomials and graphs
12. **D.12**Factor sums and differences of cubes
13. **D.13**Solve equations with sums and differences of cubes
14. **D.14**Factor using a quadratic pattern
15. **D.15**Solve equations using a quadratic pattern
16. **D.16**Pascal's triangle
17. **D.17**Pascal's triangle and the Binomial Theorem
18. **D.18**Binomial Theorem I
19. **D.19**Binomial Theorem II

Rational functions

1. **E.1**Rational functions: asymptotes and excluded values
2. **E.2**Solve rational equations
3. **E.3**Check whether two rational functions are inverses

Exponential and logarithmic functions

1. **F.1**Domain and range of exponential and logarithmic functions
2. **F.2**Convert between exponential and logarithmic form
3. **F.3**Evaluate logarithms
4. **F.4**Change of base formula
5. **F.5**Product property of logarithms
6. **F.6**Quotient property of logarithms
7. **F.7**Power property of logarithms
8. **F.8**Evaluate logarithms using properties
9. **F.9**Solve exponential equations using factoring
10. **F.10**Solve exponential equations using logarithms
11. **F.11**Solve logarithmic equations with one logarithm
12. **F.12**Solve logarithmic equations with multiple logarithms
13. **F.13**Identify linear and exponential functions
14. **F.14**Exponential functions over unit intervals
15. **F.15**Describe linear and exponential growth and decay
16. **F.16**Exponential growth and decay: word problems
17. **F.17**Compound interest: word problems

Systems of equations

1. **I.1**Solve a system of equations by graphing
2. **I.2**Solve a system of equations by graphing: word problems
3. **I.3**Classify a system of equations
4. **I.4**Solve a system of equations using substitution
5. **I.5**Solve a system of equations using substitution: word problems
6. **I.6**Solve a system of equations using elimination
7. **I.7**Solve a system of equations using elimination: word problems
8. **I.8**Solve a system of equations in three variables using substitution
9. **I.9**Solve a system of equations in three variables using elimination
10. **I.10**Determine the number of solutions to a system of equations in three variables

Systems of inequalities

1. **J.1**Solve systems of linear inequalities by graphing
2. **J.2**Solve systems of linear and absolute value inequalities by graphing
3. **J.3**Find the vertices of a solution set
4. **J.4**Linear programming